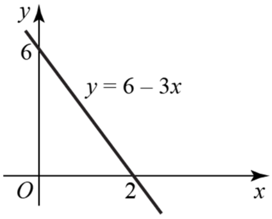
Exercise 5A

1



Mass $= M = \frac{1}{2}\rho \times 2 \times 6 = 6\rho$, where ρ is the mass per unit area.

$$M\bar{x} = \int_0^2 \rho x(6-3x)\,dx \qquad M\bar{y} = \int_0^2 \rho \frac{1}{2}(6-3x)^2\,dx$$

$$= \rho\int_0^2 6x - 3x^2\,dx \qquad = \frac{1}{2}\rho\int_0^2 36 - 36x + 9x^2\,dx$$

$$= \rho\left[3x^2 - x^3\right]_0^2 \qquad = \frac{1}{2}\rho\left[36x - 18x^2 + 3x^3\right]_0^2$$

$$= \rho[4-0] \qquad = \frac{1}{2}\rho[24-0]$$

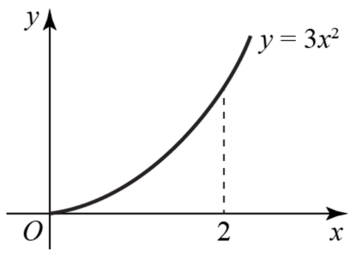
$$= 4\rho \qquad = 12\rho$$

$$\therefore \bar{x} = \frac{4\rho}{M} = \frac{4\rho}{6\rho} \qquad \therefore \bar{y} = \frac{12\rho}{M} = \frac{12\rho}{6\rho}$$

$$= \frac{2}{3} \qquad = 2$$

The centre of mass is at the point with coordinates $\left(\frac{2}{3}, 2\right)$

2



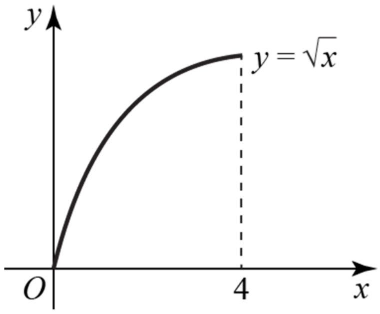
$$M = \int_0^2 \rho y \,dx$$
$$= \rho \int_0^2 3x^2 dx$$
$$= \rho\left[x^3\right]_0^2$$
$$= 8\rho$$

$$M\bar{x} = \int_0^2 \rho x 3x^2 \,dx$$
$$= \rho \int_0^2 3x^3 dx$$
$$= \rho\left[\frac{3}{4}x^4\right]_0^2$$
$$= 12\rho$$
$$\therefore \bar{x} = \frac{12\rho}{M} = \frac{12\rho}{8\rho}$$
$$= \frac{3}{2} = 1.5$$

$$M\bar{y} = \int_0^2 \frac{1}{2}\rho\left(3x^2\right)^2 dx$$
$$= \frac{1}{2}\rho \int_0^2 9x^4 \,dx$$
$$= \frac{9}{2}\rho\left[\frac{x^5}{5}\right]_0^2$$
$$= \frac{9\times 32}{10}\rho$$
$$= \frac{144}{5}\rho$$
$$\therefore \bar{y} = \frac{144\rho}{5M} = \frac{144\rho}{40\rho}$$
$$= 3.6$$

The centre of mass is at the point with coordinates (1.5, 3.6).

3



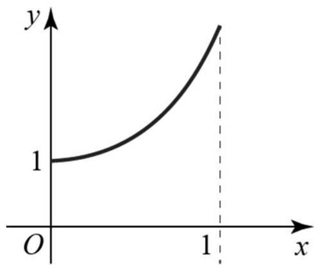
$$M = \rho\int_0^4 y\,dx$$
$$= \rho\int_0^4 x^{\frac{1}{2}}\,dx$$
$$= \left[\frac{2}{3}\rho x^{\frac{3}{2}}\right]_0^4$$
$$= \frac{16\rho}{3}$$

$$M\bar{x} = \rho\int_0^4 x x^{\frac{1}{2}}\,dx$$
$$= \rho\int_0^4 x^{\frac{3}{2}}\,dx$$
$$= \left[\frac{2}{5}\rho x^{\frac{5}{2}}\right]_0^4$$
$$= \frac{64\rho}{5}$$
$$\therefore \bar{x} = \frac{64\rho}{5M}$$
$$= \frac{64\rho}{5} \div \frac{16\rho}{3}$$
$$= \frac{64\rho}{5} \times \frac{3}{16\rho}$$
$$= \frac{12}{5} = 2.4$$

$$M\bar{y} = \frac{1}{2}\rho\int_0^4 \left(x^{\frac{1}{2}}\right)^2\,dx$$
$$= \frac{1}{2}\rho\int_0^4 x\,dx$$
$$= \frac{1}{2}\rho\left[\frac{x^2}{x}\right]_0^4$$
$$= 4\rho$$
$$\therefore \bar{y} = \frac{4\rho}{M}$$
$$= 4\rho \div \frac{16\rho}{3}$$
$$= 4\rho \times \frac{3}{16\rho}$$
$$= \frac{3}{4} = 0.75.$$

The centre of mass is at the point with coordinates (2.4, 0.75).

4



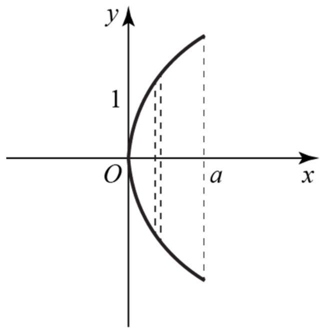
$$M = \rho\int_0^1 (x^3+1)\,dx$$
$$= \rho\left[\frac{1}{4}x^4 + x\right]_0^1$$
$$= \frac{5}{4}\rho$$

$$M\bar{x} = \rho\int_0^1 x\left(x^3+1\right)dx$$
$$= \rho\int_0^1 x^4 + x^4\,dx$$
$$= \rho\left[\frac{1}{5}x^5 + \frac{1}{2}x^2\right]_0^1$$
$$= \frac{7}{10}\rho$$
$$\therefore \bar{x} = \frac{7}{10}\rho \div M$$
$$= \frac{7}{10}\rho \div \frac{5}{4}\rho$$
$$= \frac{7\rho}{10}\times\frac{4}{5\rho}$$
$$= \frac{28}{50} = \frac{14}{25}$$

$$M\bar{y} = \frac{1}{2}\rho\int_0^1 \left(x^3+1\right)^2 dx$$
$$= \frac{1}{2}\rho\int_0^1 x^6 + 2x^3 + 1\,dx$$
$$= \frac{1}{2}\rho\left[\frac{1}{7}x^3 + \frac{1}{2}x^4 + x\right]_0^1$$
$$= \frac{\rho}{2}\times\frac{23}{14} = \frac{23\rho}{28}$$
$$\therefore \bar{y} = \frac{23\rho}{28} \div M$$
$$= \frac{23\rho}{28} \div \frac{5}{4}\rho$$
$$= \frac{23\rho}{28}\times\frac{4}{5\rho}$$
$$= \frac{23\rho}{35\rho} = \frac{23}{35}$$

The centre of mass is at the point with coordinates $\left(\frac{14}{25}, \frac{23}{35}\right)$

5



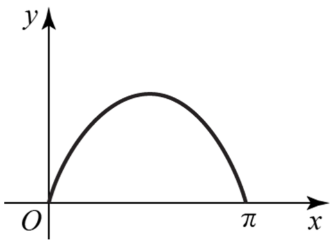
$$M = \rho\int_0^a 2y\,\mathrm{d}x$$
$$= \rho\int_0^a 2\times 2a^{\frac{1}{2}}x^{\frac{1}{2}}\,\mathrm{d}x$$
$$= 4\rho a^{\frac{1}{2}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$$
$$= \frac{8}{3}\rho a^2$$

$$M\bar{x} = \rho\int_0^a x\times 4a^{\frac{1}{2}}x^{\frac{1}{2}}\,\mathrm{d}x$$
$$= \rho\int_0^a 4a^{\frac{1}{2}}x^{\frac{3}{2}}\,\mathrm{d}x$$
$$= \rho\times 4a^{\frac{1}{2}}\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^a$$
$$= \frac{8\rho}{5}a^3$$
$$\therefore \bar{x} = \frac{8\rho a^3}{5}\div M$$
$$= \frac{8\rho a^3}{5}\div\frac{8\rho a^2}{3}$$
$$= \frac{3}{5}a$$

From symmetry $\bar{y} = 0$

$\therefore$ The centre of mass is at the point with coordinates $\left(\frac{3}{5}a, 0\right)$

6



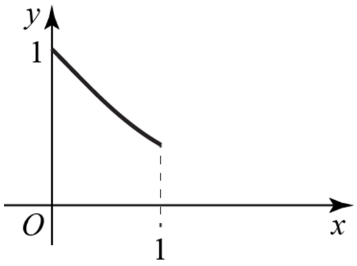
$$y = \sin x$$
$$M = \rho \int_0^{\pi} \sin x \, dx$$
$$= \rho \left[-\cos x \right]_0^{\pi}$$
$$= 2\rho$$

From symmetry $\bar{x} = \dfrac{\pi}{2}$

$$M\bar{y} = \frac{1}{2}\rho \int_0^{\pi} \sin^2 x \, dx$$
$$= \frac{1}{2}\rho \times \frac{1}{2}\int_0^{\pi} 1 - \cos 2x \, dx$$
$$= \frac{1}{4}\rho \left[x - \frac{1}{2}\sin 2x \right]_0^{\pi}$$
$$= \frac{1}{4}\rho\pi$$
$$\therefore \bar{y} = \frac{1}{4}\rho\pi \div M = \frac{1}{4}\rho\pi \div 2\rho$$
$$= \frac{1}{8}\pi$$

The centre of mass is at the point with coordinates $\left(\dfrac{\pi}{2}, \dfrac{\pi}{8} \right)$

7



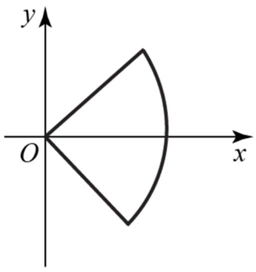
$$M = \rho\int_0^1 y\,dx$$
$$= \rho\int_0^1 \frac{1}{1+x}\,dx$$
$$= \rho\left[\ln(1+x)\right]_0^1$$
$$= \rho\ln 2$$

$$M\bar{x} = \rho\int_0^1 \frac{x}{1+x}\,dx$$
$$= \rho\int_0^1 1-\frac{1}{1+x}\,dx$$
$$= \rho\left[x-\ln(1+x)\right]_0^1$$
$$= \rho[1-\ln 2]$$
$$\therefore \bar{x} = \rho\frac{[1-\ln 2]}{M}$$
$$= \rho\frac{[1-\ln 2]}{\rho\ln 2}$$
$$= \frac{1-\ln 2}{\ln 2}$$

$$M\bar{y} = \frac{1}{2}\rho\int_0^1 \frac{1}{(1+x)}\,dx$$
$$= \frac{1}{2}\rho\left[-(1+x)^{-1}\right]_0^1$$
$$= \frac{1}{2}\rho\left[\frac{-1}{2}+1\right]$$
$$= \frac{1}{4}\rho$$
$$\therefore \bar{y} = \frac{1}{4}\rho \div M$$
$$= \frac{\frac{1}{4}\rho}{\rho\ln 2}$$
$$= \frac{1}{4\ln 2}$$

The centre of mass is at the point with coordinates $\left(\frac{1-\ln 2}{\ln 2}, \frac{1}{4\ln 2}\right)$

8



$M = \frac{1}{4}\rho\pi r^2$ as this is a quarter of a circle.

$$M\bar{x} = \rho\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2 \times \frac{2}{3}r\cos\theta\,\mathrm{d}\theta$$

$$= \rho \times \frac{1}{3}r^3\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{3}\rho r^3\left[\sin\theta\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{3}\rho r^3\left[\frac{1}{\sqrt{2}} - \left(\frac{-1}{\sqrt{2}}\right)\right]$$

$$= \frac{2\rho r^3}{3\sqrt{2}}$$

$$\therefore \bar{x} = \frac{2\rho r^3}{3\sqrt{2}} \div \frac{1}{4}\rho\pi r^2$$

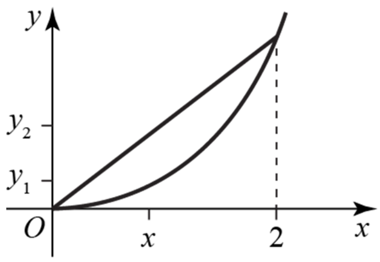
$$= \frac{8r}{3\pi\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{8\sqrt{2}}{6\pi} = \frac{4\sqrt{2}}{3\pi}$$

Also $\bar{y} = 0$ from symmetry.

The centre of mass is at the point with coordinates $\left(\frac{4\sqrt{2}r}{3\pi}, 0\right)$

9



$y = x^3$ meets $y = 4x$ when $x^3 = 4x$ i.e. $x = \pm 2$.
$\therefore$ when $x > 0, x = 2$

The small strip shown has dimensions $(y_2 - y_1)$ by δx and centre of mass at $\left(x, \frac{1}{2}(y_1 + y_2)\right)$

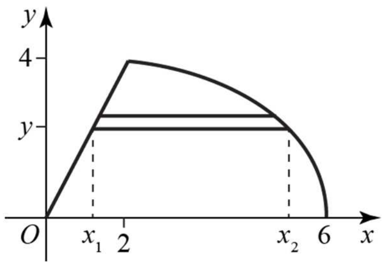
$$M = \rho\int_0^2 (4x - x^3)\,\mathrm{d}x$$
$$= \rho\left[2x^2 - \frac{1}{4}x^4\right]_0^2$$
$$= 4\rho$$

$$M\bar{x} = \rho\int_0^2 x(4x - x^3)\,\mathrm{d}x$$
$$= \rho\int_0^2 4x^2 - x^4\,\mathrm{d}x$$
$$= \rho\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$
$$= \rho\left[\frac{32}{3} - \frac{32}{5}\right]$$
$$= \frac{64\rho}{15}$$
$$\therefore \bar{x} = \frac{64\rho}{15} \div M = \frac{64\rho}{15} \div 4\rho$$
$$= \frac{16}{15}$$

$$M\bar{y} = \frac{1}{2}\rho\int_0^2 (y_1 + y_2)(y_2 - y_1)\,\mathrm{d}x$$
$$= \frac{1}{2}\rho\int_0^2 (4x + x^3)(4x - x^3)\,\mathrm{d}x$$
$$= \frac{1}{2}\rho\int_0^2 16x^2 - x^6\,\mathrm{d}x$$
$$= \frac{\rho}{2}\left[\frac{16}{3}x^3 - \frac{1}{7}x^7\right]_0^2$$
$$= \frac{\rho}{2}\left[\frac{128}{3} - \frac{128}{7}\right]$$
$$= \frac{256}{21}\rho$$
$$\therefore \bar{y} = \frac{256}{21}\rho \div M = \frac{256}{21}\rho \div 4\rho$$
$$= \frac{64}{21}$$

The centre of mass is at the point with coordinates $\left(\frac{16}{15}, \frac{64}{21}\right)$

10



Divide the region into horizontal strips of dimensions $(x_2 - x_1)$ by δy.

The centre of mass of such strips lies at $\left(\frac{x_1 + x_2}{2}, y\right)$

where $x_1 = \frac{y}{2}$ and $x_2 = \frac{24 - y^2}{4}$

Using $M = \rho\int_0^4 (x_2 - x_1)\,dy$

$$\therefore M = \rho\int_0^4 \left(6 - \frac{1}{4}y^2 - \frac{y}{2}\right)dy$$

$$= \rho\left[6y - \frac{1}{12}y^3 - \frac{1}{4}y^2\right]_0^4$$

$$= \rho\left[24 - \frac{16}{3} - 4\right]$$

$$= \frac{44}{3}\rho$$

Using $M\bar{y} = \rho\int_0^4 y(x_2 - x_1)\,dy$

$$M\bar{y} = \rho\int_0^4 6y - \frac{1}{4}y^3 - \frac{y^2}{2}\,dy$$

$$= \rho\left[3y^2 - \frac{1}{16}y^4 - \frac{1}{6}y^3\right]_0^4$$

$$= \rho\left[48 - 16 - \frac{64}{6}\right]$$

$$= \frac{64}{3}\rho$$

$$\therefore \bar{y} = \frac{64}{3}\rho \div M = \frac{64}{3} \times \frac{3}{44}$$

$$= \frac{16}{11}$$

$$= 1.5 \text{ (2 s.f.)}$$

10 continued

Using $M\bar{x} = \rho\int_0^6 yx\,\mathrm{d}x = \rho\int_0^2 yx\,\mathrm{d}x + \rho\int_2^6 yx\,\mathrm{d}x$

$$= \rho\int_0^2 2x^2\,\mathrm{d}x + \rho\int_2^6 x\sqrt{24-4x}\,\mathrm{d}x$$

$$= \rho\left[\frac{2}{3}x^3\right]_0^2 - \rho\left[\frac{4}{5}(x+4)(6-x)\sqrt{6-x}\right]_2^6$$

$$= \rho\left[\frac{16}{3} + \frac{192}{5}\right] = \rho\left[\frac{656}{15}\right]$$

$$\therefore\ \bar{x} = \frac{656}{15}\rho \div M = \frac{656}{15} \times \frac{3}{44}$$

$$= \frac{164}{55}$$

$$= 2.98\,(2\text{ s.f.})$$

Hence the centre of mass has coordinates $\left(\frac{164}{55}, \frac{16}{11}\right)$

11 By symmetry, the centre of mass of the uniform lamina will lie on the $y = 0$ axis. The mass of the lamina is $M = 2\int_0^4 \rho y\,\mathrm{d}x$, where ρ is the density. Integrating gives $M = 2\rho\int_0^4 \frac{x+5}{(x+2)(2x+1)}\,\mathrm{d}x$

$$= 2\rho\int_0^4 \left(\frac{3}{2x+1} - \frac{1}{x+2}\right)\mathrm{d}x$$

$$= 2\rho\left[\frac{3}{2}\ln(2x+1) - \ln(x+2)\right]_0^4$$

$$= 2\rho\big(\ln(27) - \ln(3)\big) = 2\rho\ln(9)$$

Now using the formula $M\,\bar{x} = 2\int_0^4 \rho xy\,\mathrm{d}x$,

$$M\,\bar{x} = 2\rho\int_0^4 \frac{(x+5)x}{(x+2)(2x+1)}\,\mathrm{d}x$$

$$= 2\rho\int_0^4 \left(-\frac{3}{2(2x+1)} + \frac{2}{x+2} + \frac{1}{2}\right)\mathrm{d}x$$

$$= 2\rho\left[-\frac{3}{4}\ln(2x+1) + 2\ln(x+2) + \frac{1}{2}x\right]_0^4$$

$$= 2\rho\left(-\tfrac{1}{4}\ln(27) + 2\ln(3) + 2\right) = \rho\big(4 + \ln(3)\big)$$

Hence $\bar{x} = \dfrac{\rho\big(4+\ln(3)\big)}{4\rho\ln(3)} \approx 1.16\,(3\text{ s.f.})$, and $\bar{y} = 0$

12 The mass of the lamina is $M=\int_0^3 \rho y\,\mathrm{d}x=\rho\int_0^3 \frac{1}{\sqrt{x^2+4}}\,\mathrm{d}x$

$=\rho\left[\operatorname{arcsinh}\left(\frac{1}{2}x\right)\right]_0^3=\rho\operatorname{arcsinh}\left(\frac{3}{2}\right)$

Using the formula $M\bar{x}=\int_0^3 \rho xy\,\mathrm{d}x$

$=\rho\int_0^3 \frac{x}{\sqrt{x^2+4}}\,\mathrm{d}x$. Make a change of variable $u=x^2+4$, $\mathrm{d}u=2x\,\mathrm{d}x \Rightarrow$

$M\bar{x}=\rho\int_4^{13} \frac{1}{2\sqrt{u}}\,\mathrm{d}u=\rho\left[\sqrt{u}\right]_4^{13}=\rho\left(\sqrt{13}-2\right)$.

We also have $M\bar{y}=\int_0^3 \rho\frac{1}{2}y^2\,\mathrm{d}x$

$=\rho\int_0^3 \frac{1}{x^2+4}\,\mathrm{d}x=\frac{1}{4}\rho\left[\arctan\left(\frac{1}{2}x\right)\right]_0^3$

$=\frac{1}{4}\rho\arctan\left(\frac{3}{2}\right)$. Hence,

$\bar{x}=\frac{\left(\sqrt{13}-2\right)}{\operatorname{arcsinh}\left(\frac{3}{2}\right)}\approx 1.34$ (3 s.f.), and $\bar{y}=\frac{\arctan\left(\frac{3}{2}\right)}{4\operatorname{arcsinh}\left(\frac{3}{2}\right)}\approx 0.206$ (3 s.f.)

Challenge

First we find the centre of mass of the pendant. The equation of the pendant curve is $x=y^2$.

Because the pendant is symmetric, $\bar{y}=0$. Using the formula $M\bar{x}=2\int_0^4 \rho xy\,\mathrm{d}x$

$=2\rho\int_0^4 x\sqrt{x}\,\mathrm{d}x=2\rho\left[\frac{2}{5}x^{5/2}\right]_0^4=\rho\frac{128}{5}$. The mass of the pendant is $M=2\int_0^4 \rho y\,\mathrm{d}x$

$=2\rho\int_0^4 \sqrt{x}\,\mathrm{d}x=2\rho\left[\frac{2}{3}x^{3/2}\right]_0^4=\rho\frac{32}{3}$, hence $\bar{x}=2.4$. If the pendant is suspended at any point P on its perimeter, no part of the pendant should be higher than P. If we connect P to the centre of mass with a line ℓ_1, and draw a line ℓ_2 at P which is perpendicular to ℓ_1, no parts of pendant should be above that line. Some possible points are immediately obvious: these are on the symmetry axis $y=0$, $(0,0)$ and $(4,0)$. Suppose another such point is (x_0,y_0). The equation for ℓ_1 is $\frac{y-\bar{y}}{x-\bar{x}}=-\frac{1}{k}$, and for ℓ_2 is $\frac{y-y_0}{x-x_0}=k=\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x_0}=\frac{1}{2\sqrt{x_0}}$. Hence, at (x_0,y_0) we have $\frac{y_0-\bar{y}}{x_0-\bar{x}}=-2\sqrt{x_0}$. Note that $y_0=\sqrt{x_0}$.

Now we can find $x_0=1.9$, and $y_0=\sqrt{1.9}$. By symmetry, point $\left(1.9,-\sqrt{1.9}\right)$ will also be a solution. We also need to check the edges, where the tangent to the pendant is not defined. This gives us two more points $(4,\pm 2)$.